Convergent Solutions to the First-Order Difference Equation

ALFRED J. SUROWIEC

North American Utility Construction Company, New York, New York

A recent communication (*1*) notes the uncertainty of convergence of the series solution given by Haldane (*2*) for the first-order difference equation

$$y_n - y_{n+1} = \phi\,(y_{n+1}) \tag{1}$$

There is an alternate solution to this equation in which convergence is easily ascertained. The first two terms in this solution, identical to those found above, arose in a natural way in the study of multicomponent distillation (*4*).

$$n \cong -\int_o^n \frac{dy_{n+1}}{y_n - y_{n+1}} - \frac{1}{2}\int_o^n \frac{d(y_n - y_{n+1})}{(y_n - y_{n+1})} \tag{2}$$

A Taylor series expansion (*5*) yielded a third term:

$$-\frac{1}{12}\int_o^n \frac{1}{(y_n - y_{n+1})}\, d\left[\frac{d(y_n - y_{n+1})}{dn}\right] \tag{3}$$

and an alternating remainder series

$$\int_o^n \frac{dn}{(y_n - y_{n+1})} \sum_{k=2}^{\infty} (-a_{2k}) \frac{d^{2k}(y_n - y_{n+1})}{dn^{2k}} \tag{4}$$

In most cases, Equation (2) is a close approximation and it can be expanded by induction (*6*) and used to evaluate the third term given by Equation (3). When this is done, the resulting infinite series has first and second terms which are identical to the third and fourth terms of the Haldane solution (*3*), while the coefficient of the next term is 15/720 instead of 19/720 as required. An equivalent form that is simpler to use is given by

$$-\frac{1}{12}\int_o^n \frac{1}{(y_n - y_{n+1})}\, d\left[\frac{d(y_n - y_{n+1})}{dn}\right] \cong -\frac{1}{6}\int_o^n d\,A_n - \frac{1}{6}\int_o^n A_n\, d\ln\,(y_n - y_{n+1}) \tag{5}$$

The second integral on the right-hand side of the above expression can be estimated by taking an average value of

$$A_n = \frac{1 - \dfrac{dy_n}{dy_{n+1}}}{1 + \dfrac{dy_n}{dy_{n+1}}};\quad (A_n)_{av.} = \left(\frac{A_o + A_n}{2}\right) \tag{6}$$

If the difference $y_n - y_{n+1}$ is a decreasing function of n and its derivatives are either strictly increasing or strictly decreasing functions of n, then the sum of terms in Equation (4) is a convergent alternating series such that

$$\int_o^n \frac{dn}{(y_n - y_{n+1})} \sum_{k=2}^{\infty} (-a_{2k}) \frac{d^{2k}(y_n - y_{n+1})}{dn^{2k}} < \frac{n(\ln r_\infty)^4}{720} \tag{7}$$

where

$$r_\infty = \lim_{n\to\infty} \left(\frac{y_n - y_{n+1}}{y_{n-1} - y_n}\right) \tag{8}$$

Thus, an approximate solution is available for the first-order difference equation which permits the estimation of the degree of approximation and the determination of convergence.

The usefulness of this solution is shown in Table 1 by a comparison with some of the results of Davison for the solution of the equation

$$y_n - y_{n+1} = y_{n+1}\,(K + y_{n+1}) \tag{9}$$

TABLE 1. COMPARISON OF SOLUTIONS OF EQUATION (9)

y_1	y_{n+1}	K	n	No. of terms in Equation (16), reference *1*: 2	No. of terms in Equation (16), reference *1*: 5	Equations (2), (5), and (6)	Equation (7)
0.100	0.0147	0.2	9	9.01	8.98	8.99	0.000014
0.912	0.100	0.2	6	6.22	5.77	5.97	0.000009
0.765	0.0737	0.6	4	4.18	3.74	3.94	0.00003
1.138	0.0111	0.8	7	7.47*	5.11	7.01	0.0012
1.015	0.260	−0.2	8	8.20	7.80	7.94	0.000027

* Recalculated.

LITERATURE CITED

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